UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

LEVEL II

(2)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER: AFOSR-TR- 81 -0370	2. GOVT ACCESSION NO: AD-A098063	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle): Convergence of adaptive minimum variance algorithms via Weighting Coefficient Selection.		5. TYPE OF REPORT & PERIOD COVERED: INTERIM rept.
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s): Rajendra kumar ~~and~~ John B. Moore		8. CONTRACT OR GRANT NUMBER(s): AFOSR-76-3063, NSF-ENG77-12946
9. PERFORMING ORGANIZATION NAME AND ADDRESS: Lefschetz Center for Dynamical Systems, Division of Applied Mathematics, Brown University, Providence, R. I. 02912		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS: 2304/A4, 61102F
11. CONTROLLING OFFICE NAME AND ADDRESS: AFOSR/NM, Bolling AFB, DC 20332	TR-EE-7917	12. REPORT DATE: August 1979
		13. NUMBER OF PAGES: 33
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office): Feb 81		15. SECURITY CLASS. (of this report): Unclassified 35
		15a. DECLASSIFICATION DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

DTIC ELECTE APR 22 1981 E

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

See abstract attached.

411824

DD FORM 1 JAN 73 1473 EDITION OF 1 NOV 65 IS OBSOLETE

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

UNCLASSIFIED

ABSTRACT

Weighted least squares, and related stochastic approximation algorithms are studied for parameter estimation, adaptive state estimation, adaptive N-step-ahead prediction, and adaptive control, in both white and coloured noise environments. For the fundamental algorithm which is the basis for the various applications, the step size in the stochastic approximation versions and the weighting coefficient in the weighted least squares schemes are selected according to a readily calculated stability measure associated with the estimator. The selection is guided by the convergence theory. In this way, strong global convergence of the parameter estimates, state estimates, prediction or tracking errors is not only guaranteed under the appropriate noise, passivity, and stability or minimum phase conditions, but also the convergence is as fast as it appears reasonable to achieve given the simplicity of the adaptive scheme.

AFOSR-TR- 81-0370

CONVERGENCE OF ADAPTIVE MINIMUM VARIANCE ALGORITHMS VIA WEIGHTING COEFFICIENT SELECTION †

by

Rajendra Kumar*

and

John B. Moore+

Technical Report No. EE 7917

August, 1979

Revised July, 1980, February, 1981

---

† Work supported by the Australian Research Grant Committee. Also, supported in part by the Air Force Office of Scientific Research under Contract #AFOSR-76-3063 and in part by the National Science Foundation under contract #NSF-Eng. 77-12946 A02.

* Was with Department of Electrical Engineering, University of Newcastle. Presently with Division of Applied Mathematics, Brown University, Providence, Rhode Island 02912.

+ Department of Electrical Engineering, University of Newcastle, New South Wales, 2308, Australia.

ABSTRACT

Weighted least squares, and related stochastic approximation algorithms are studied for parameter estimation, adaptive state estimation, adaptive N-step-ahead prediction, and adaptive control, in both white and coloured noise environments. For the fundamental algorithm which is the basis for the various applications, the step size in the stochastic approximation versions and the weighting coefficient in the weighted least squares schemes are selected according to a readily calculated stability measure associated with the estimator. The selection is guided by the convergence theory. In this way, strong global convergence of the parameter estimates, state estimates, prediction or tracking errors is not only guaranteed under the appropriate noise, passivity, and stability or minimum phase conditions, but also the convergence is as fast as it appears reasonable to achieve given the simplicity of the adaptive scheme.

1. INTRODUCTION

Based on the very simple ideas of least squares parameter estimation and stochastic approximation it is not difficult to propose adaptive estimators, predictors and controllers which work quite well in a range of environments. However, existing schemes sometimes behave poorly in the absence of persistency of excitation of the state estimates. In the presence of instabilities, or what appears to be instability over a short time period, this lack of "persistence" may cause divergence of parameter estimates. To pin down the precise conditions under which such schemes work well is of considerable interest. For adaptive control, the task is made difficult since it is unreasonable to add any a priori assumptions concerning the closed-loop system stability.

A key objective of this paper is to demonstrate for linear stochastic signal models that it is possible for the theory to guide in the design of the adaptive algorithms so as to ensure parameter and/or prediction error convergence with the convergence rate being as fast as appears reasonable to achieve with simple adaptive schemes.

In earlier work [1, 2], the convergence of least squares and extended least squares stochastic adaptive schemes are studied using a martingale convergence theory. A sufficient condition of crucial importance, exposed in this theory, is that a system related to the signal generating system or frequently just the noise generating system be passive (or have a positive real transfer function in the time invariant linear signal model case). Simulation studies and the theory of [3, 4] also suggest that this condition is close to being a necessary one. Also fundamental to the parameter convergence theory of [1, 2] is a persistence of excitation condition. The

theory does not exclude the possibility of instabilities arising in closed-loop control which cause lack of persistence of excitation in some modes, and thus divergence of parameter estimates, compounded by ill-conditioned calculations. Least squares results have also been reported in [3], and in [4] via an ordinary differential equation approach [5]. The work of [1-5], without modification falls short of giving a global convergence analysis for adaptive control.

More recently in [6], a specific adaptive control scheme has been proposed for which global strong convergence results are derived without any a priori stability assumptions. The theory builds on the martingale approach, and on the earlier deterministic theory of [7-10]. However, our simulation experience shows that the performance is inferior (e.g. 100 times slower) to that of the self-tuning schemes of [11-14] when these converge. These self-tuning control schemes use least squares ideas but in common with the schemes of [1, 2], their convergence theory requires a priori assumptions about their stability. This is not fully satisfactory in a control situation.

An attempt to generalize the stochastic approximation approach of [6] by harnessing the power of a least squares approach is given in [15]. In this work, a stability measure is taken to be a bound on the condition number of this estimation error "covariance" matrix employed in the least squares approach. When a somewhat arbitrary bound on this number is exceeded, then the algorithm uses a stochastic approximation scheme tailored to the error "coveriance" matrix at the switching time. The scheme uses a priori prediction error estimates in the state estimator and is dramatically inferior to the schemes of [6] for some coloured noise applications. A revised version of [15] translates ideas from the technical report [16],

the antecedant of the present paper, to treat the case of a posteriori prediction errors in the state estimator. In one scheme it employs the stability measure of [16].

In this paper, we tolerate a reduced weighting in the extended least squares performance index when there is what appears to be insufficient excitation assessed over a finite time period. This reduced weighting overrides any other weighting scheme such as "exponential weighting" applied for the initial transient period. As a consequence, global convergence results are achieved for an algorithm near in some sense to the standard extended least squares scheme. The convergence results here are stronger than in the revised version of [15], giving convergence rates and also than in the technical report [16] on which this paper is based.

The specific contributions of the paper are summarized as follows. The first contribution is to give a global convergence theory for weighted least squares schemes and related stochastic approximation schemes. The important by-product of this contribution is to give a simple scheme for weighting coefficient selection to ensure global convergence in closed-loop adaptive control. At the heart of the weighting coefficient selction schemes is a persistance of excitation/ stability measure already available in the calculations. The second contribution is to show how earlier global convergence theory results of [6, 15, 16] can be strengthened to give convergence of the prediction errors, to the appropriate white noise term. In contrast to earlier work, convergence rates are implicit in the present theory, as are convergence rates for parameter estimate differences. Another distinctive feature of the theory of the present paper is an implicit lower bound on the convergence rate of the prediction or tracking

error to the white noise term even in the absence of persistency of excitation. A third contribution is to show that under persistently exciting conditions, zero bias parameter convergence is established. A final contribution is to show how the theory can be generalized for N-step-ahead prediction/control schemes without using an interleaved bank of parameter estimators as in [17].

In Section 2, the weighted extended least squares algorithm is introduced and in Section 3, its global convergence properties are studied. In Section 4, the case of N-step-ahead prediction/control is considered and in Section 5, some concluding remarks are made.

2. SIGNAL MODELS, ADAPTIVE ALGORITHMS AND CONVERGENCE CONDITIONS

Signal Model Class. Consider the signal model

$$z_k = \theta' x_k + \nu_k \tag{2.1}$$

where z_k is the measurement p-vector sequence, and x_k is the state n-vector, and θ is the unknown $n \times p$ parameter matrix. The noise ν_k is a zero mean white process or more precisely is assumed to satisfy for some σ_ν

$$E[\nu_k | F_{k-1}] = 0, \quad E[\| \nu_k \|^2 | F_{k-1}] \le \sigma_\nu^2 \tag{2.2a}$$

$$\limsup_{k\to\infty} \frac{1}{k} \sum_{0}^{k} \| \nu_i \|^2 < \infty \tag{2.2b}$$

where F_k denotes the minimal σ-algebra generated by $\nu_0, \nu_1, \ldots, \nu_k$, $x_0, x_1, \ldots, x_k$, $z_0, z_1, \ldots, z_k$ and θ.

If the states x_k of such a model are known, then parameter estimation can be achieved in terms of x_k. Otherwise, a standard approach is to replace x_k by an estimate $\hat{x}_k$. For this we need a more specific description of a model for x_k. Consider the state model

$$x_{k+1} = (F + G_1\theta')x_k + G_2\nu_k + f(u_k, z_k) \tag{2.3}$$

where F, G_1, G_2, $f(\cdot, \cdot)$ are known, possibly time-varying functions.

This model (2.1)-(2.3) encompasses a number of useful special cases. For example, it covers the autoregressive moving average model class with exogenous inputs (ARMAX) of the form, given here for the scalar measurement case as

$$z_k = \sum_{i=1}^{n} a_i z_{k-i} + \sum_{i=1}^{m} b_i u_{k-i} + \sum_{i=1}^{r} c_i \nu_{k-i} + \nu_k \tag{2.4}$$

with exogenous inputs u_k. Details in [2] are not repeated here save that we can define

$$x_k' = [z_{k-1} \dots z_{k-n}\ u_{k-1} \dots u_{k-m}\ \nu_{k-1} \dots \nu_{k-r}]$$

$$\theta' = [a_1 \dots a_n\ b_1 \dots b_m\ c_1 \dots c_r]$$

It is also true that (2.3) encompasses multivariable ARMAX models (with unit delay) in which the various scalar parameters and scalar variables are replaced by matrices and vectors respectively. That such models can be used to represent a very general class of linear systems with unit delay is shown in [18]. We are not here constrained to a unique representation, or a minimal representation of a multivariable system, although such models are of course preferable for some applications.

Another example of the class of model encompassed by (2.1)-(2.3) is the transfer function model class where for polynomials A, B, C, D, in the delay operator q^{-1}, $z_k = BA^{-1}u_k + CD^{-1}\nu_k$. Details are omitted here.

The model (2.1)-(2.3) also has application to adaptive Kalman filtering where a co-ordinate basis is specified, as discussed in [2].

<u>State Estimation</u>. With the state space model (2.1)-(2.3), given some estimate $\hat{\theta}_k$ at time k, a state estimator is

$$\hat{x}_{k+1} = F\hat{x}_k + G_1\hat{y}_k + G_2\tilde{z}_{k/k} + f_k(u_k, z_k) \tag{2.5a}$$

$$\tilde{z}_{k/k} = z_k - \hat{y}_k,\ \hat{y}_k = \hat{\theta}_k'\hat{x}_k \tag{2.5b}$$

For the scalar ARMAX model (2.4),

$$\hat{x}_k' = [z_{k-1} \dots z_{k-n}\ u_{k-1} \dots u_{k-m}\ \tilde{z}_{k-1/k-1} \dots \tilde{z}_{k-r/k-r}]$$

<u>State Estimation Error Equations</u>. From (2.3)-(2.5) we have

$$\tilde{x}_{k+1} = (F + G\theta')\tilde{x}_k + G(\tilde{\theta}_k'\hat{x}_k) \tag{2.6}$$

where $G = G_1 - G_2$ and $\tilde{x}_k = x_k - \hat{x}_k$. Observe that $f(\cdot, \cdot)$ does not influence (2.6).

Parameter Estimation. The global convergence theory of the next section requires that parameter estimates are given from

$$\hat{\theta}_k = \hat{\theta}_{k-1} + \hat{B}_{k-1}\hat{x}_k(\hat{\gamma}_k^{-1} + \hat{x}_k'\hat{B}_{k-1}\hat{x}_k)^{-1}\tilde{z}_{k/k-1}' \tag{2.7a}$$

$$\tilde{z}_{k/k-1} = z_k - \hat{\theta}_{k-1}'\hat{x}_k \tag{2.7b}$$

where $\hat{B}_k > 0$ satisfies

$$\hat{B}_{k-1}^{-1} \le \hat{B}_k^{-1} \le \hat{B}_{k-1}^{-1} + \hat{\gamma}_k\hat{x}_k\hat{x}_k' \tag{2.8}$$

For weighted least squares versions, then for some $\hat{B}_0 >> 0$, $\hat{\gamma}_k > 0$

$$\hat{B}_k = \hat{B}_{k-1} - \hat{B}_{k-1}\hat{x}_k\hat{x}_k'\hat{B}_{k-1}(\hat{\gamma}_k^{-1} + \hat{x}_k'\hat{B}_{k-1}\hat{x}_k)^{-1}$$

$$\hat{\gamma}_k\hat{B}_k\hat{x}_k = (\hat{\gamma}_k^{-1} + \hat{x}_k'\hat{B}_{k-1}\hat{x}_k)^{-1}\hat{B}_{k-1}\hat{x}_k, \quad \hat{B}_{k-1}\hat{x}_k = (1-\hat{\gamma}_k\hat{x}_k'\hat{B}_k\hat{x}_k)^{-1}\hat{B}_k\hat{x}_k \tag{2.9}$$

(Recall that for standard least squares, $\hat{\gamma}_k$ is a constant $\hat{\gamma}$ and $\hat{\gamma}\hat{B}_k = (\sum_0^k \hat{x}_i\hat{x}_i')^{-1}$ which is independent of $\hat{\gamma}$.) For stochastic approximation, $\hat{B}_k = \hat{B}_{k-1} = B > 0$. Typically $B = I$ and $\hat{\gamma}_k = (\sum_0^k \hat{x}_i'\hat{x}_i)^{-1}$. The actual $\hat{\gamma}_k$ selection is specified later in this section.

Parameter Estimation Error. Defining $\tilde{\theta}_k = \theta - \hat{\theta}_k$, then from (2.7), (2.5b), (2.9), simple manipulations yield

$$\tilde{z}_{k/k-1} = \theta'\tilde{x}_k + \tilde{\theta}_{k-1}'\hat{x}_k + \nu_k, \quad \tilde{z}_{k/k} = \theta'\tilde{x}_k + \tilde{\theta}_k'\hat{x}_k + \nu_k \tag{2.10a}$$

$$\tilde{z}_{k/k-1} = (1 + \hat{\gamma}_k\hat{x}_k'\hat{B}_{k-1}\hat{x}_k)\tilde{z}_{k/k} = (1-\hat{\gamma}_k\hat{x}_k'\hat{B}_k\hat{x}_k)^{-1}\tilde{z}_{k/k} \tag{2.10b}$$

$$\tilde{\theta}_k = \tilde{\theta}_{k-1} - \frac{\hat{B}_{k-1}\hat{x}_k}{\hat{\gamma}_k^{-1} + \hat{x}_k'\hat{B}_{k-1}\hat{x}_k}\tilde{z}_{k/k-1}' = \tilde{\theta}_{k-1} - \hat{\gamma}_k\hat{B}_{k-1}\hat{x}_k\tilde{z}_{k/k}' \tag{2.11}$$

<u>Minimum Variance Control</u>. We consider the case when the plant output z_k is to be controlled to track a specified trajectory z_k^*. In minimum variance control, by choosing the control so that the one-step-ahead prediction estimate $\hat{z}_{k+1/k}$ is the specified trajectory z_{k+1}^*, then the tracking error is the one-step-ahead prediction error, which is of course "minimized" in a least squares sense by the state and parameter estimation procedures. Thus the control u_k is selected so that the implicit equations

$$\hat{\theta}_k' \hat{x}_{k+1}(u_k) = z_{k+1}^* \tag{2.12}$$

are satisfied.

An explicit solution for u_k is readily found and is often unique. For example, for the ARMAX model (2.4), if $\hat{b}_1 \neq 0$, at some time k then u_k is unique and simply calculated. If $b_1 = 0$ as when there is a known delay $N > 1$ in the plant, then N step-ahead prediction may be called for. This is studied in Section 4.

<u>Passivity Condition</u>. It is known [2], that the parameter estimation error equations and state estimation error equations can be organized* as a feedforward subsystem with states $\tilde{x}_k$ and a feedback subsystem with states $\tilde{\theta}_k$, and an external noise input ν_k. Moreover, the feedback system, in the appropriate organization, turns out to be passive, as defined in [20]. Since, it is known that a strictly passive system back to back with a passive system has input/output stability behaviour for its subsystems [20], it is not surprising that in the convergence theory of the next section, one of a set of sufficient conditions is that the feedforward subsystem be strictly passive. The relevant feedforward subsystem is

$$\xi_{k+1} = (F + G\theta')\xi_k + Gq_k, \; p_k = \theta'\xi_k + \tfrac{1}{2}q_k, \; q_k \stackrel{\Delta}{=} \tilde{\theta}_k' \hat{x}_k \tag{2.13}$$

*The details are included in remark 4, following the proof of Theorem 3.1.

and we require that this be input strictly passive, and output strictly passive, or equivalently for some $\kappa \geq 0$, $\varepsilon > 0$, and all m

$$\sum_{0}^{m} q_k'(p_k - \varepsilon q_k) \geq -\kappa, \quad \sum_{0}^{m} p_k'(q_k - \varepsilon p_k) \geq -\kappa \tag{2.14}$$

For the time invariant case, an equivalent condition is that

$$\{\tfrac{1}{2}I + \theta'[zI-(F+G\theta')]^{-1}G\} =$$
$$\{[I - \theta'(zI - F)^{-1}G]^{-1} - \tfrac{1}{2}I\} \text{ is strictly positive real [21]} \tag{2.15}$$

and for the ARMAX specialization,* (see Appendix for proof)

$$[C^{-1}(z) - \tfrac{1}{2}] \text{ is strictly positive real [21]} \tag{2.16}$$

where $C(z) = 1 + C_1 z^{-1} + \cdots C_r z^{-r}$

. The conditions (2.14) have the interpretation of a passivity condition for a system with input q_k and output $(p_k - \varepsilon q_k)$, and for a system with input p_k and output $(q_k - \varepsilon p_k)$ respectively. The theorem of [20, page 178] tells us that passive systems followed by a monotonically decreasing gain are also passive. So that here with $\gamma_k > 0$ monotonically decreasing, a consequence of the above passivity condition (2.14) is that for some $\kappa > 0$, $\varepsilon > 0$ and all m.

$$\sum_{0}^{m} q_k'\gamma_k(p_k - \varepsilon q_k) \geq -\kappa, \quad \sum_{0}^{m} p_k'\gamma_k(q_k - \varepsilon p_k) \geq -\kappa \tag{2.17}$$

A discrete-time version of this derivation is in the Appendix.

*For the scheme of [6], the condition is $[C^{-1}(z) - \frac{a}{2}]$ is strictly positive real where $0<a\leq 1$ is a scale factor on the parameter update step size. There does not appear to be a corresponding simplification here.

<u>Bounds</u>. For open-loop, one-step-ahead prediction error convergence to the noise ν_k, it is not surprising that a bound on the plant output and states is required. Such a priori restrictions are intolerable for closed-loop adaptive control but they have their parallel in order to keep the control signals bounded. The parallel conditions are the same as for the case when the plant parameters are known, namely that the desired trajectory z_k^* be bounded and that the plant be minimum phase, or more precisely, that the plant have a bounded-state, bounded-input, and a bounded-output, bounded-state property. Equivalently, the plant inverse system must have a bounded-input, bounded-state and a bounded-state, bounded-output property, which is guaranteed if it is exponentially asymptotically stable and is uniformly completely observable and reachable (see discrete-time versions of results in [22]). Thus we introduce

<u>Open-loop Prediction-Bounds</u>. For some κ and all $m > m_1$ for some m_1

$$\frac{1}{m}\sum_{0}^{m} \| x_k \|^2 \leq \kappa, \tag{2.18}$$

<u>Closed-loop Adaptive Control-Bounds</u>. For some κ and all $m > m_1$ for some m_1

$$\frac{1}{m}\sum_{0}^{m} \| z_k^* \|^2 \leq \kappa \tag{2.19}$$

$$\frac{\bar{\kappa}}{m}\sum_{0}^{m} \| u_k \|^2 \leq \frac{1}{m}\sum_{0}^{m} \| x_k \|^2 \leq \frac{\kappa}{m}\sum_{0}^{m} \| z_k \|^2 + \kappa \tag{2.20}$$

This latter condition is referred to as a <u>"minimum phase" condition</u>.

For the ARMAX model and noise restriction (2.2), then (2.20) holds trivially for the case of adaptive control with $B(z) = b_1 z^{-1} + b_2 z^{-2} + \ldots b_m z^{-m}$ minimum phase, or equivalently with all zeros within the disc $|z| < 1$. In contrast, (2.18) holds if $A(z) = 1+a_1 z^{-1} + \ldots a_n z^{-n}$ is minimum phase and u_k is bounded in L_2. Other cases are not spelled out in detail.

Weighting Coefficient Selection. For the one step ahead prediction algorithm, we make the following selection of $\hat{\gamma}_k$ with $\alpha_k \triangleq \hat{x}_k' \hat{B}_{k-1} \hat{x}_k$ and $\varepsilon > 0$ some small number,

$$\bar{\gamma}_k = \begin{cases} 1 & \text{if } k \in \mathcal{S}_1 \triangleq \{k: \alpha_k \leq \frac{\bar{K}}{k}\} & (2.21a) \\ (k\alpha_k)^{-1/2} & \text{if } k \in \mathcal{S}_2 \triangleq \{k: \alpha_k \leq K,\ k \notin \mathcal{S}_1\} & (2.21b) \\ C\gamma_k^{-1/2} & \text{if } k \in \mathcal{S}_3 \triangleq \{k: k \notin \mathcal{S}_1, \mathcal{S}_2\} & (2.21c) \end{cases}$$

$$\bar{K} >> 0, \quad 0 < C, K < \infty, \quad r_k \triangleq \sum_{j=0}^{k} \| \hat{x}_j \|^2, \quad \bar{r}_k = \max(k, r_k) \qquad (2.21d)$$

$$\hat{\gamma}_k = \min\{\bar{\gamma}_k, \hat{\gamma}_{k-1}\}, \ \gamma_k = \hat{\gamma}_k \delta_k, \ \delta_k = \bar{r}_k^{-\varepsilon} \qquad (2.21e)$$

For the adaptive control algorithm the above selection is used except that (2.21e) is replaced by the following.

$$\hat{\gamma}_k = \min\{\bar{\gamma}_k \bar{r}_k^{-\varepsilon}, \hat{\gamma}_{k-1}\}, \ \gamma_k = \hat{\gamma}_k \delta_k, \ \delta_k \equiv 1 \qquad (2.21e')$$

The selection of $\hat{\gamma}_k$ is made so as to satisfy certain summability conditions for the application of mastingale convergence theorem and thereby derive the result that tr $\{\tilde{\theta}_k' \hat{B}_k^{-1} \tilde{\theta}_k \delta_k\}$ converges and $\sum_0^\infty \gamma_k \| \tilde{z}_{k/\bar{k}} \nu_k \|^2 < \infty$ a.s. The monotone nature of γ_k has already been alluded to in the discussion of passivity condition, essentially it keeps the passivity condition same as for the standard least square algorithm. Additionally, this selection satisfies the condition $\liminf_{k\to\infty} \gamma_k \bar{r}_k^{(1/2+\varepsilon)} \geq K_2 > 0$. This condition ensures that under a minimum phase restriction on the plant, closed loop system stability is achieved for the minimum variance controller. These properties of $\hat{\gamma}_k$ are summarized in the following lemma, the proof of which is given in the appendix.

<u>Lemma 2.1</u> The $\hat{\gamma}_k, \delta_k$ selections in (2.21) satisfy the following.

$$\sum_{k=0}^{\infty} \delta_k \hat{\gamma}_k^2 \hat{x}_k' \hat{B}_{k-1} \hat{x}_k \leq K_1 < \infty \tag{2.22a}$$

$$\liminf_{k\to\infty} \gamma_k \bar{r}_k^{-(1/2+\varepsilon)} \geq K_2 > 0 \tag{2.22b}$$

$$\hat{\gamma}_k \hat{x}_k' \hat{B}_{k-1} \hat{x}_k \leq K_3 < \infty , \quad k \in \mathscr{S}_1, \mathscr{S}_2 , \quad K_3 \geq \|\hat{B}_0\| \tag{2.22c}$$

<u>Remarks.</u> 1. The distinction between the prediction algorithm and the adaptive control algorithm has been necessitated due to the fact that whereas $\hat{\gamma}_k = 1$ is a possibility for the prediction algorithm, the analysis technique of this paper does not permit such selection for the adaptive control algorithm, unless additional assumptions like persistency of excitation are made. Of course, the $\hat{\gamma}_k$ selection involving (2.21e') could be used for both prediction and control applications.

2. For the prediction algorithm, if after a finite time $\bar{k}$, α_k satisfies $\alpha_k \leq \frac{\bar{K}}{k}$, then $\hat{\gamma}_k$ becomes a constant after such time. Thus the algorithm behaves as standard least square algorithm under persistency of excitation .

3. Under minimum phase restriction on the plant for adaptive control algorithm and stability restriction for the open loop prediction algorithm, it will be shown in the subsequent section that $\hat{\gamma}_k$ is bounded from below by $k^{-1/2}$ without requiring any persistency of excitation condition .

4. The $\bar{\gamma}_k$ selection could be simplified by eliminating (2.21b, without changing the convergence analysis of the subsequent sections.

3. CONVERGENCE ANALYSIS

The convergence results are presented as two major theorems, now stated and proved.

Theorem 3.1: Consider the plant (2.1)-(2.3) and the weighted extended least squares estimation schemes of Section 2 with the $\hat{\gamma}_k, \delta_k, \gamma_k$ satisfying (2.21). Then with the plant constrained by the Passivity Condition of Section 2,

(i) $\limsup_{k\to\infty} \mathrm{tr}\{\tilde{\theta}_k' \hat{B}_k^{-1} \tilde{\theta}_k\} \delta_k < \infty$ a.s.

$\lim_{k\to\infty} \delta_k^{-1} \hat{B}_k = 0 \implies \lim_{k\to\infty} \hat{\theta}_k = \theta$ a.s.

(ii) $\sum_0^\infty ||\hat{B}_{k-1}||^{-1} \delta_k ||\hat{\theta}_k - \hat{\theta}_{k-j}||^2 < \infty$ a.s. for all finite j.

(iii) $\sum_0^\infty \gamma_k \| \tilde{z}_{k/k} - \nu_k \|^2 < \infty$ a.s.

$\sum_0^\infty \gamma_k \| \tilde{x}_k \|^2 < \infty$ a.s.

In addition, for the adaptive control algorithm, one also obtains

(iv) $\sum_0^\infty \tau_k \| \tilde{z}_{k/k-1} - \nu_k \|^2 < \infty$ a.s.

or $\sum_0^\infty \tau_k \| \tilde{z}_k^* - \nu_k \|^2 < \infty$ a.s.

where $\tau_k = \gamma_k$ if $k \in \mathcal{S}_1, \mathcal{S}_2$ and $\tau_k = \gamma_k r_k^{-1/2+\varepsilon}$ otherwise

Proof: Result (i). First define for $\varepsilon > 0$ and κ given in (2.17)

$$V_k = \mathrm{tr}\{\tilde\theta_k'\hat B_k^{-1}\tilde\theta_k\}\delta_k + \{2\sum_0^k[\gamma_k p_k' q_k - \varepsilon\gamma_k(\|q_k\|^2 + \|p_k\|^2)] + \kappa\} \tag{3.1}$$

where $p_k = \theta'\tilde x_k + \frac{1}{2}\tilde\theta_k'\hat x_k$, $q_k = \tilde\theta_k'\hat x_k$. Observe that $V_k \geq 0$ by virtue of (2.17), guaranteed by the Passivity Condition. Simple manipulations now yield,

$$E[V_k|F_{k-1}] \leq V_{k-1} + E[\hat\Delta_k - 2\gamma_k\nu_k' q_k|F_{k-1}] - \varepsilon\gamma_k E[(\|q_k\|^2 + \|p_k\|^2)|F_{k-1}] \tag{3.2}$$

where

$$\hat\Delta_k = \mathrm{tr}\{\tilde\theta_k'\hat B_k^{-1}\tilde\theta_k - \tilde\theta_{k-1}'\hat B_{k-1}^{-1}\tilde\theta_{k-1} + 2\hat\gamma_k(p_k+\nu_k)'q_k\}\delta_k$$

$$= \mathrm{tr}\{\tilde\theta_k'(\hat B_k^{-1} - \hat B_{k-1}^{-1})\tilde\theta_k - \hat\gamma_k^2(\hat x_k'\hat B_{k-1}\hat x_k)\tilde z_{k/k}'\tilde z_{k/k} - \hat\gamma_k\tilde\theta_k'\hat x_k q_k'\}\delta_k$$

The latter equality follows from a substitution for $\tilde\theta_{k-1}$ from (2.11b). Applying the inequality (2.8) and definition for q_k gives that $\hat\Delta_k = -\eta_k$ where η_k is defined below. Also, from the definition of q_k and (2.11a)

$$E[2\gamma_k\nu_k' q_k|F_{k-1}] = 2\gamma_k E[\nu_k'\tilde\theta_k'\hat x_k|F_{k-1}] = -\beta_k$$

$$\beta_k \triangleq 2\gamma_k(\hat x_k'\hat B_{k-1}\hat x_k)(\hat\gamma_k^{-1} + \hat x_k'\hat B_{k-1}\hat x_k)^{-1}E[\|\nu_k\|^2|F_{k-1}] \tag{3.3}$$

with

$$\eta_k \triangleq \delta_k\hat\gamma_k^2\hat x_k'\hat B_{k-1}\hat x_k\|\tilde z_{k/k}\|^2$$

Application of these results in (3.2) gives

$$E[V_k|F_{k-1}] \leq V_{k-1} - \varepsilon\gamma_k E[\|q_k\|^2 + \|p_k\|^2|F_{k-1}] + \beta_k - \eta_k \tag{3.4}$$

or with $\hat V_k = V_k + \varepsilon\gamma_k(\|q_k\|^2 + \|p_k\|^2) + \eta_k$

$$E[\hat V_k|F_{k-1}] \leq \hat V_{k-1} - \varepsilon\gamma_{k-1}(\|q_{k-1}\|^2 + \|p_{k-1}\|^2) + \beta_k - \eta_{k-1} \tag{3.5}$$

The martingale convergence theorem {pages 33,[23]}, now tells us that for arbitrary $\hat{V}_k \geq 0$, $\beta_k \geq 0$, $\sum_0^\infty \beta_k < \infty$, then almost surely $\hat{V}_k$ converges and $\sum_0^\infty \gamma_k \| q_k \|^2 < \infty$, $\sum_0^\infty \gamma_k \| p_k \|^2 < \infty$. Application here is straight forward since the strict passivity condition ensures that $\hat{V}_k \geq 0$, and the γ_k selection to ensure (2.22a) and the noise restriction (2.2 a) ensures that $\sum_0^\infty \beta_k < \infty$. Thus under the conditions of the theorem

$$\sum_0^\infty \gamma_k \| \tilde{z}_{k/k} - \nu_k \|^2 = \sum_0^\infty \gamma_k \| p_k + \tfrac{1}{2} q_k \|^2 \leq \tfrac{1}{2} \sum_0^\infty \gamma_k (\| q_k \|^2 + 4 \| p_k \|^2) < \infty \text{ a.s.}$$

$$\sum_0^\infty \delta_k \hat{\gamma}_k^{\,2} \hat{x}_k' \hat{B}_{k-1} \hat{x}_k \| \tilde{z}_{k/k} \|^2 < \infty \text{ a.s.} \tag{3.6}$$

Also $\hat{V}_k$ converges almost surely.

The first part of result (i) of the theorem follows since the additive terms comprising $\hat{V}_k$ are all positive and thus for each, denoted $\hat{V}_k^{(i)}$, $\limsup_{k\to\infty} \hat{V}_k^{(i)} < \infty$. The second part of (i) follows from the first part.

<u>Result (ii)</u>. From (2.11)

$$\delta_k \|\hat{B}_{k-1}\|^{-1} \|\hat{\theta}_k - \hat{\theta}_{k-1}\|^2 \leq \delta_k \hat{\gamma}_k^{\,2} x_k' \hat{B}_{k-1} \hat{x}_k \| \tilde{z}_{k/k} \|^2 \tag{3.7}$$

The result thus follows from second part of (3.6).

<u>Result (iii)</u>. Part (a) is merely (3.6) which has thus been established. Result (iii b) follows since the system (2.13) rewritten as

$$\gamma_{k+1}^{\frac{1}{2}} \tilde{x}_{k+1} = [\gamma_{k+1}^{\frac{1}{2}} (F+G\theta') \gamma_k^{-\frac{1}{2}}] \gamma_k^{\frac{1}{2}} \tilde{x}_k + (G \gamma_{k+1}^{\frac{1}{2}} \gamma_k^{-\frac{1}{2}}) \gamma_k^{\frac{1}{2}} q_k$$

has a bounded-input, bounded-state property by virtue of its asymptotic stability following from the strict positive real condition on (2.13), as in (2.15), and the fact that γ_k monotonically decreases from (2.21e). A crucial intermediate step is

$$\sum_0^k || \gamma_{k+1}^{\frac{1}{2}} (F+G\theta')^{k-i} \gamma_i^{-\frac{1}{2}} || \leq \sum_0^k || F+G\theta' ||^{k-i} < \infty$$

for some κ and all k, otherwise the arguments are standard.

To obtain the result (iv) for adaptive control, premultiplying (2.11) by $\hat{x}_k'$ and taking its norm yields,

$$\hat{\gamma}_k \| \hat{x}_k'(\hat{\theta}_k - \hat{\theta}_{k-1}) \|^2 \leq \hat{\gamma}_k^3 (\hat{x}_k' \hat{B}_{k-1} \hat{x}_k)^2 \| \tilde{z}_{k/k} \|^2$$

Applying (2.22c), noting that $\delta_k \equiv 1$, for the adaptive control algorithm and the application of second inequality of (3.6) results in

$$\tau_k \| \hat{x}_k'(\hat{\theta}_k - \hat{\theta}_{k-1}) \|^2 \leq K_3 \delta_k \hat{\gamma}_k^2 \hat{x}_k' \hat{B}_{k-1} \hat{x}_k \quad \| \tilde{z}_{k/k} \|^2$$

and thus $$\sum_0^{\infty} \tau_k \| \hat{x}_k'(\hat{\theta}_k - \hat{\theta}_{k-1}) \|^2 < \infty \quad \text{a.s.} \tag{3.8}$$

<u>Result (iv a)</u> follows from (iii a) and (3.8) since

$$\| \tilde{z}_{k/k-1} - \nu_k \|^2 \leq 2 \| \tilde{z}_{k/k} - \nu_k \|^2 + 2 \| \hat{x}_k' (\hat{\theta}_k - \hat{\theta}_{k-1}) \|^2 .$$

Result (iv b) simply designates that in adaptive control, the tracking error $\tilde{z}_k^*$ is equal to $\tilde{z}_{k/k-1}$.

∇∇∇

<u>Remark</u> 1. In terms of the parameter and state estimation errors, the prediction and the adaptive control algorithms are alike. However, for the prediction algorithm it has not been possible to deduce the convergence of a priori prediction error $\tilde{z}_{k/k-1}$ from that of the a posteriori prediction error $\tilde{z}_{k/k}$. This is due to the fact

that $\tilde{z}_{k/k-1} = (1 + \hat{\gamma}_k \hat{x}_k' \hat{B}_{k-1} \hat{x}_k) \tilde{z}_{k/k}$ and the $\hat{\gamma}_k$ selection does not ensure any a priori bound on $\hat{\gamma}_k \hat{x}_k' \hat{B}_{k-1} \hat{x}_k$. The presence of the factor $\bar{r}_k^{-\varepsilon}$ in $\hat{\gamma}_k$ for the control algorithm just enables to obtain the result that

$$\lim_{k \to \infty} \sup \frac{1}{k} \sum_0^k \| x_i \|^2 < \infty.$$

(See also next theorem.)

2. The passivity condition is automatically satisfied for the case when $\hat{x}_k = x_k$ is known as for example for ARMAX models with white noise. Simulations and the theory of [4,5] suggest that otherwise it is close to being a necessary condition. The condition was first exploited in [1-4] and is not discussed further here.

3. The result (i) is a generalization of [1, 2] for the closed loop estimation using weighted least squares algorithm. The results (ii), (iii) are novel and have no correspondance to earlier results for the case $\hat{\gamma}_k = 1$.

4. As noted in section 2, the state and parameter estimation error equations can be re-organized as a feedforward system with states $\tilde{x}_k$ back to back with a feedback system with states $\tilde{\theta}_k$ as

$$\tilde{x}_{k+1} = (F+G\theta')\tilde{x}_k + Gq_k \ , \ p_k = \theta'\tilde{x}_k' + \tfrac{1}{2}q_k$$

$$\tilde{\theta}_k = \tilde{\theta}_{k-1} - \hat{\gamma}_k\hat{B}_{k-1}(p_k + \tfrac{1}{2}q_k + \nu_k)' \ , \ q_k = \tilde{\theta}_k'\hat{x}_k$$

The result $\hat{\Delta}_k \leq 0$ in the proof of result (i) above leads to the conslusion, via a passivity theorem also in [1,2], that the linear time varying feedback system with inputs $(p_k+\nu_k)$ and output $(\hat{\gamma}_k q_k)$ is passive. Since δ_k is monotonically decreasing, the same system but with output $\delta_k\hat{\gamma}_k q_k = \gamma_k q_k$ is also passive.

5. An additional cosntraint namely the Minimum Phase/ Stability condition of Section 2, can be introduced t guarantee bounded signals and lower bounds on γ_k as follows.

<u>Theorem 3.2</u>: Under the conditions of Theorem 3-1 and with the Minimum Phase/Stability conditions of Section 2, holding, then the estimation schemes of Section 2 yield the bounds

$$\lim\sup \frac{1}{k}\sum_{0}^{k}\|\hat{x}_i\|^2 < \infty \qquad \text{a.s.} \tag{3.9a}$$

$$\liminf_{k\to\infty} \gamma_k\, k^{(1/2+\varepsilon)} > 0, \quad \lim_{k\to\infty} \hat{\gamma}_k \hat{x}_k' \hat{B}_{k-1} \hat{x}_k = 0 \quad \text{a.s.} \tag{3.9b}$$

$$\sum_{0}^{\infty} k^{-(1/2+\varepsilon)}\|\tilde{x}_k\|^2 < \infty \qquad \text{a.s.} \tag{3.9c}$$

$$\lim_{k\to\infty} \frac{1}{k}\sum_{0}^{k}\|\tilde{x}_i\|^2 = 0 \qquad \text{a.s.} \quad (\text{convergence rate } k^{1/2}) \tag{3.9c}$$

$$\sum_{0}^{\infty} k^{-(1/2+\varepsilon)}\|\tilde{z}_{k/k-1} - \nu_k\|^2 < \infty, \quad \lim_{k\to\infty} k^{-(1/2+\varepsilon)}\sum_{0}^{k}\|\tilde{z}_{i/i-1} - \nu_i\|^2 = 0 \quad \text{a.s.} \tag{3.9d}$$

In addition, for the adaptive control algorithm

$$\sum_{0}^{\infty} k^{-(1/2+\varepsilon)}\|\tilde{z}_k^* - \nu_k\|^2 < \infty, \quad \lim_{k\to\infty} k^{-(1/2+\varepsilon)}\sum_{0}^{k}\|\tilde{z}_i^* - \nu_i\|^2 = 0 \quad \text{a.s.} \tag{3.9e}$$

$$\limsup_{k\to\infty} \frac{1}{k}\sum_{0}^{k}\|x_i\|^2 < \infty, \quad \limsup_{k\to\infty} \frac{1}{k}\sum_{0}^{k}\|u_i\|^2 < \infty, \quad \text{a.s.}$$

$$\limsup_{k\to\infty} \frac{1}{k}\sum_{0}^{k}\|z_i\|^2 < \infty \quad \text{a.s.} \tag{3.9f}$$

<u>*Proof*</u>: Considering first the prediction algorithm, exploiting the lower bound on $\hat{\gamma}_k$ of (2.22b), and the Kronecher lemma [23], then result (iii) of Theorem 3.1 yields

$$\lim_{k\to\infty} [\max k, r_k]^{-(1/2+\varepsilon)} \sum_{i=0}^{k}\|\tilde{x}_i\|^2 = 0 \quad \text{a.s.} \tag{3.10}$$

Now since $\|\hat{x}_i\|^2 \le 2\|x_i\|^2 + 2\|\tilde{x}_i\|^2$, the application of (2.18) results in

$$\frac{1}{k}\sum_{0}^{k}\|\hat{x}_i\|^2 \le \frac{\kappa}{k}\sum_{0}^{k}\|\tilde{x}_i\|^2 + \kappa \tag{3.11}$$

for some κ, $0 < \kappa < \infty$. Simple manipulations then imply (3.9a). In brief, in view of (3.11)

$$[\max k, \sum_0^k \| \hat{x}_i \|^2]^{-(1/2+\varepsilon)} \sum_0^k \| \tilde{x}_i \|^2 \geq \{\frac{\kappa}{k}\sum_{i=0}^k \| \tilde{x}_i \|^2 + (\kappa+1)\}^{-(1/2+\varepsilon)} \times \frac{1}{k}\sum_{i=0}^k \| \tilde{x}_i \|^2$$

This inequality implies that $\limsup_{k\to\infty} \frac{1}{k}\sum_0^k \| \tilde{x}_i \|^2 < \infty$ a.s.

for otherwise taking limits for a subsequence there is the contradiction that $0 \geq \infty$. This bound in (3.11) establishes (3.9a) for the prediction algorithm. For the adaptive control algorithm the lower bound on τ_k namely $\tau_k \bar{r}_k \geq \bar{K}_2$, for some $\bar{K}_2 < \infty$ from (2.22b) the Kronecher lemma [23], and result (iv) of Theorem 3.1 yield

$$\lim_{k\to\infty} [\max k, r_k]^{-1} \sum_{i=0}^k \|\tilde{z}_{i/i-1} - \nu_i\|^2 = 0 \quad \text{a.s.}$$

For the adaptive control algorithm, it will be established in the sequal that,

$$\frac{1}{k}\sum_0^k \| \hat{x}_i \|^2 \leq \frac{\kappa}{k}\sum_{i=0}^k \| \tilde{z}_{i/i-1} - \nu_i \|^2 + \kappa \tag{3.12}$$

These two inequalities establish (3.9a) for the adaptive control algorithm by repeating the above argument verbatim. This along with (2.22b) of Lemma 2.1 implies the first part of (3.9b). That (3.9a) implies the second part of (3.9b) follows from the lemma A2 of [17] in view of the boundedness of $\hat{\gamma}_k$. Now the first part of (2.10b) implies that

$$\|\tilde{z}_{k/k-1} - \nu_k \|^2 \leq 2 \|\tilde{z}_{k/k} - \nu_k \|^2 + 2(\hat{\gamma}_k \hat{x}_k' B_{k-1} \hat{x}_k)^2 \| \tilde{z}_{k/k} \|^2$$

The application of the result (iii) of Theorem 3.1, second part of (3.9b) and the second part of (3.6) imply that $\sum_0^{\infty} \gamma_k \|\tilde{z}_{k/k-1} - \nu_k\|^2 < \infty$ a.s. The result (3.9d) then follows by applying the first part of (3.9b) and the Kronecker lemma. Result (3.9e) holds as a consequence of (3.9d) since for the adaptive control algorithm, the tracking error $\tilde{z}_k^*$ equals the prediction error $\tilde{z}_{k/k-1}$. Also, $\|x_k\|^2 \leq 2\|\hat{x}_k\|^2 + 2\|\tilde{x}_k\|^2$, and the first part of (3.9f) follows from (3.9a) and (3.9c). The second part of (3.9f) is a consequence of the minimum phase condition (2.20) while the rest follows from (2.1) and (2.2b).

It remains to show that (3.12) follows from the theorem assumptions. A consequence of the minimum phase condition (2.20) and the bounds (2.2b), (2.19) in the following inequality

$$\frac{1}{k}\sum_0^k \|z_i\|^2 \leq 3\frac{1}{k}\sum_0^k \|\tilde{z}_i^* - \nu_i\|^2 + 3\frac{1}{k}\sum_0^k \|z_i^*\|^2 + 3\frac{1}{k}\sum_0^k \|\nu_i\|^2$$

gives for some κ

$$\frac{1}{k}\sum_0^k \|x_i\|^2 \leq \frac{\kappa}{k}\sum_0^k \|\tilde{z}_i^* - \nu_i\|^2 + \kappa = \frac{\kappa}{k}\sum_0^k \|\tilde{z}_{i/i-1} - \nu_i\|^2 + \kappa \qquad (3.13)$$

Also, $\|\tilde{x}_i\|^2$ is bounded in terms of $\|p_i + \tfrac{1}{2}q_i\|^2 = \|\theta'\tilde{x}_i + \tilde{\theta}_i'\hat{x}_i\|^2 = \|\tilde{z}_{i/i} - \nu_i\|^2$ as in the passivity condition statement, and $\|\tilde{z}_{i/i} - \nu_i\|^2 \leq 2\|\tilde{z}_{i/i}\|^2 + 2\|\nu_i\|^2 \leq 2\|\tilde{z}_{i/i-1}\|^2 + 2\|\nu_i\|^2 \leq 4\|\tilde{z}_{i/i-1} - \nu_i\|^2 + 6\|\nu_i\|^2$, giving that $\|\tilde{x}_i\|^2$ is bounded in terms of $\|\tilde{z}_{i/i-1} - \nu_i\|^2$ and $\|\nu_i\|^2$. Now since $\|\hat{x}_i\|^2 \leq 2\|x_i\|^2 + 2\|\tilde{x}_i\|^2$, application of (3.13) and this bound gives (3.12).

∇∇∇

Remarks 1. The results (3.9c,d) of Theorem 3.2 and their antecedent results (iii, iv) of Theorem 3.1 are stronger than the previous results [6, 15] which do not give explicit convergence of $\tilde{z}_{i/i-1}$ to ν_i, or the implicit convergence rates. Here, in addition to the convergence of $\tilde{z}_{i/i-1}$ to ν_i, an implicit lower bound of $i^{1/2}$ on the convergence rate of the prediction/tracking error and the state estimation error is established independent of any persistency condition.

2. It is easily seen that if the matrix $\hat{B}_k$ or $(\frac{1}{k}\sum_{i=0}^{k} x_i x_i')$ decreases at a rate $\frac{1}{k}$ or faster then $\hat{\gamma}_k$ will remain nearly constant for the prediction algorithm and nearly $k^{-\varepsilon}$ for the adaptive control algorithm. This is in view of the result that $\limsup_{k\to\infty} \frac{1}{k}\sum_0^k \hat{x}_i \hat{x}_i' < \infty$. However as the bound is not uniform, it is not possible to conclude that $\hat{\gamma}_k$ will exactly be a constant under such conditions.

3. It remains an open question as to whether or not $\|\tilde{x}_k\|$ can fail to converge to zero without a weighting coefficient selection and in the absence of persistency of excitation.

4. The techniques of this paper as well as those of [6, 15] prove the various convergence in the Cesaro sense. In a subsequent paper [28], the uniform boundedness of $\hat{x}_k$ and the uniform convergence of prediction error is established for a related algorithm using the projection methods of [29, 30].

4. N-STEP AHEAD ADAPTIVE PREDICTION/CONTROL SCHEMES

If there is a delay of N units between the application of a control signal to a plant and any response to that signal, then in controlling that plant, it makes sense to work with an N-step ahead prediction of the output of the plant. In minimum variance N-step-ahead control, the measurements $\{z_k\}$ are predicted N steps ahead as $\hat{z}_{k+N/k}(u_k)$, being expressed as a known function of the control signal at time u_k. The control u_k can then be chosen so that this prediction is the desired N-step ahead output trajectory z^*_{k+N}. That is, u_k is chosen to satisfy

$$z^*_{k+N} = \hat{z}_{k+N/k}(u_k) \tag{4.1}$$

Thus for N-step-ahead prediction/control, consider a state space model encorporating an N-delay as

$$x_{k+N} = Fx_{k+N-1} + G_1\theta' x_k + G_2 n_k + f(u_k, z_k) \tag{4.2a}$$

$$z_k = \theta' x_k + n_k,\ n_k = w_k + Q_0 w_{k-1} + \dots Q_{N-2} w_{k+1-N} \tag{4.2b}$$

with w_k satisfying (2.2).

Consider the scalar input/output model $Az_{k+N} = Bu_k + Cw_{k+N}$ with A, B, C polynomial operators in the delay operator q^{-1} with degrees n, m, p, where A, C are monic. With the long divisions $C = AF + q^{-N}G$, $1 = C\bar{F} + q^{-N}\bar{G}$ defining $F, \bar{F}$ monic and of degree $(N-1)$ and $G, \bar{G}$ monic with degrees $(n-1),(p-1)$, and definition $n_k = Fw_k$, $z_k = y_k + n_k$, then simple manipulations yield

$$Cy_{k+N} = FBu_k + Gz_k,\ y_{k+N} = \bar{G}y_k + \bar{F}FBu_k + \bar{F}Gz_k$$

Defining, for the scalar variable case

$$x'_{k+N} = [\underbrace{z_k\ z_{k-1}\ \cdots}_{n+N-1}\ \underbrace{u_k\ u_{k-1}\ \cdots}_{m+2N-1}\ \underbrace{y_k\ y_{k-1}\ \cdots}_{p}]$$

then a re-organization as in (4.2) is straightforward. Using the theory of [18], the multivariable case can be covered likewise.

Also, transfer function models can be organized as in (4.2).

With the above model (4.2), an adaptive estimator is

$$\hat{x}_{k+N} = F\hat{x}_{k+N-1} + G_1\hat{\theta}'_k\hat{x}_k + G_2\tilde{z}_{k/k} + f_k(u_k,\ z_k) \tag{4.3}$$

where $\hat{\theta}_k$ is calculated as in (2.7) - (2.9) in terms of $\hat{x}_k$ of (4.3). A prediction $\hat{z}_{k+N/k} = \hat{\theta}'_k\hat{x}_{k+N}$, and control u_k is selected to satisfy $z^*_{k+N} = \hat{\theta}'_k\hat{x}_{k+N}(u_k)$.

With the definitions $q_k = (\tilde{\theta}'_k\hat{x}_k)$, $p_k = (\frac{1}{2}\tilde{\theta}'_k\hat{x}_k+\theta'\tilde{x}_k)$ as earlier, then (4.2), (4.3) yield

$$\tilde{x}_{k+N} = F\tilde{x}_{k+N-1} + G(p_k + \tfrac{1}{2}q_k),\ p_k = \theta'\tilde{x}_k + \tfrac{1}{2}q_k \tag{4.4}$$

and the passivity condition of interest is as follows.

<u>Passivity Convergence Condition</u>. The system with state equation (4.4) is output and input strictly passive, or equivalently in the time invariant case

$$\{[I - \theta' z^{-(N-1)}(zI - F)^{-1}G]^{-1} - \tfrac{1}{2}I\} \text{ is strictly positive real}$$

For the input/output model this condition is that, see also [17].

$$\{[I - z^{-(N-1)}\bar{G}(z)]^{-1} - \tfrac{1}{2}\} \text{ is strictly positive real.}$$

<u>Convergence Analysis</u>. To generalize the theory of this paper to the N-step-ahead prediction scheme above, a crucial step is to view $\tilde{x}_k$, $\tilde{\theta}_k$, as decomposed with N fictitious values as

$\tilde{x}_k^{(1)} \ldots \tilde{x}_k^{(N)}$, $\tilde{\theta}_k^{(1)} \ldots \tilde{\theta}_k^{(N)}$ with the properties $\tilde{x}_k = \sum_{i=1}^{N} \tilde{x}_k^{(i)}$, $\tilde{\theta}_k = \sum_{i=1}^{N} \tilde{\theta}_k^{(i)}$ for all k, where

$$\tilde{\theta}_k^{(i)} = \tilde{\theta}_{k-1}^{(i)} + \hat{\gamma}_k \hat{B}_k \hat{x}_k \tilde{z}_{k/k-1}^{(i)} \tag{4.5a}$$

$$\tilde{z}_{k/k-1}^{(i)} = \theta' \tilde{x}_k^{(i)} + \tilde{\theta}_{k-1}^{(i)\prime} \hat{x}_k + Q_{i-2} w_{k+1-i}, \quad Q_{-1} = I \tag{4.5b}$$

and

$$\tilde{x}_{k+N}^{(i)} = F \tilde{x}_{k+N-1}^{(i)} + G(\theta' \tilde{x}_k^{(i)} + \tilde{\theta}_k^{(i)\prime} \hat{x}_k) \tag{4.6}$$

Exploiting the fact that $\hat{x}_k$, $\hat{\gamma}_k$, $\hat{B}_k$, $\tilde{x}_k$, $\delta_k \subset F_{k-N} \subset F_{k-i}$ for $i = 1, 2 \ldots N$ and thus $\tilde{\theta}_{k-1}^{(i)} \subset F_{k-i}$, then $V_k^{(i)} = \mathrm{tr}\{\tilde{\theta}_k^{(i)\prime} \hat{B}_k^{-1} \tilde{\theta}_k^{(i)}\} \delta_k$ has the property $E[V_{k-1}^{(i)} | F_{k-i}] = V_{k-1}^{(i)}$, $E[w_{k+1-i} | F_{k-i}] = 0$. Working with $E[V_k^{(i)} | F_{k-i}]$, rather than $E[V_k^{(i)} | F_{k-1}]$, then the earlier analysis approach yields that under the passivity condition above, for $i = 1, 2, \ldots N$.

$$\limsup_{k\to\infty} V_k^{(i)} < \infty \text{ a.s.}, \quad \sum_0^\infty \gamma_k [\| \hat{x}_k' \tilde{\theta}_k^{(i)} \|^2 + \| \tilde{x}_k^{(i)\prime} \theta \|^2] < \infty \quad \text{a.s.} \tag{4.7}$$

Also, $\sum_0^\infty \tau_k \| \hat{x}_k' (\tilde{\theta}_k^{(i)} - \tilde{\theta}_{k-N}^{(i)}) \|^2 < \infty$ a.s., and consequently

$$\sum_0^\infty \tau_k \| \tilde{z}_{k/k-N}^{(i)} - Q_{i-2} w_{k+1-i} \|^2 < \infty \quad \text{a.s.} \tag{4.8}$$

From (4.7) and (4.8), and application of the triangle inequality, then

$$\limsup_{k\to\infty} \mathrm{tr}\{\tilde{\theta}_k' \hat{B}_k^{-1} \tilde{\theta}_k\} \delta_k < \infty, \quad \sum_0^\infty \tau_k \| \tilde{z}_{k/k-N} - n_k \|^2 < \infty \text{ a.s.} \tag{4.9}$$

and the global convergence results for one-step-ahead-prediction control of the previous section apply for the N-step-ahead prediction/control case with ν_k replaced by n_k, and $\tilde{z}_{k|k}$ replaced by $\tilde{z}_{k|k-N}$. τ_k in (4.8) and (4.9) can be replaced by $k^{-(1/2+\varepsilon)}$ as in the proof of Theorem 3.2.

Remarks 1. The predictors above are simpler than those in [17] involving a bank of N interlaced parameter estimators. Here, the decomposition of θ into $\theta^{(1)} \ldots \theta^{(N)}$ etc. is purely a construct in the convergence theory with no consequences for implementation. Also, the results are mildly simpler than in [19] upon which this section is based.

2. The above results are also applicable to the problem of establishing convergence in the presence of colored noise but where the state estimates are uncorrelated with the noise. A second important application is for output-error schemes which use a parallel model to achieve a state estimate uncorrelated with the noise. A subsequent paper studies this case in detail [27].

5. CONCLUSIONS

The paper has presented a weighted least squares approach in parameter estimation, N-step-ahead prediction and control. The weightings are selected according to a stability measure and guided by a global convergence theory. A feature of the approach is that we achieve open-loop adaptive prediction and identification results, and with the same theory, closed-loop adaptive control results. Thus, in our adaptive control schemes, under persistently exciting conditions, there is consistent parameters estimation, and asymptotically optimal state estimation achieved while tracking. Also, the results have application to the adaptive control of general linear nonminimum phase plants [25] and output error recursions in colored noise [27].

Appendix

<u>Proof:</u> That (2.13) specializes as $[C^{-1}(z)-\frac{1}{2}]$ for the ARMAX model (2.4). The system required to be strictly passive is one where $q_k = (\tilde{\theta}_k'\hat{x}_k)$ and $p_k = (\frac{1}{2}\tilde{\theta}_k'\hat{x}_k+\theta'\tilde{x}_k)$. Simple manipulations yield $(p_k+\frac{1}{2}q_k) = \tilde{\theta}_k'\hat{x}_k + \theta'\tilde{x}_k = \tilde{z}_{k/k} - \nu_k$ and for the ARMAX model (2.4), $(p_k-\frac{1}{2}q_k) = \theta'\tilde{x}_k = \sum_{i=1}^{r} \hat{c}_i \; (\nu_{i-1}-\tilde{z}_{i-1/i-1})$ is seen to be the output of a system $C(z)$ driven by $(p_k+\frac{1}{2}q_k)$. Standard manipulations then yield that the system with input q_k and output p_k is $[C^{-1}(z)-\frac{1}{2}]$.

<u>Proof:</u> That (2.17) follows from (2.14). In simplified notation, with $\sum_0^m \bar{p}_k'\bar{q}_k > -\kappa$ for some $\kappa \geq 0$ and all m, and $\gamma_k > 0$ $(\gamma_{-1} \equiv 0)$ monotonically decreasing, then $\sum_0^m \gamma_k\bar{p}_k'\bar{q}_k = \gamma_m \sum_{k=0}^{m} \bar{p}_k'\bar{q}_k - \sum_{k=0}^{m+1} (\gamma_k-\gamma_{k-1}) \sum_{i=0}^{k-1} \bar{p}_k'\bar{q}_k \geq -\gamma_m\kappa + \sum_{k=0}^{m+1} (\gamma_k-\gamma_{k-1})\kappa = -\gamma_0\kappa$. Here the first equality is from summation by parts, and the inequality follows from the assumptions. Thus $\sum_0^m \gamma_k\bar{p}_k'\bar{q}_k \geq -\gamma_0\kappa$. ∇∇∇

<u>Proof of Lemma</u> 2.1. Noting that $\gamma_k \leq \bar{\gamma}_k\bar{r}_k^{-\varepsilon}$,

$$\sum_{k\in\mathcal{S}_1} \hat{\gamma}_k^2\delta_k\hat{x}_k'\hat{B}_{k-1}\hat{x}_k \leq \sum_0^\infty \bar{K}\, k^{-(1+\varepsilon)} < \infty \tag{A1}$$

$$\sum_{k\in\mathcal{S}_2} \hat{\gamma}_k^2\delta_k\hat{x}_k'\hat{B}_{k-1}\hat{x}_k \leq \sum_0^\infty k^{-(1+\varepsilon)} < \infty \tag{A2}$$

$$\sum_{k\in\mathcal{S}_3} \hat{\gamma}_k^2\delta_k\hat{x}_k'\hat{B}_{k-1}\hat{x}_k \leq \sum_0^\infty \frac{\hat{x}_k'\hat{x}_k}{r_k^{(1+\varepsilon)}} \;\|\hat{B}_0\|\, C^2 < \infty \tag{A3}$$

where the last inequality follows from the lemma A1. Thus (2.22a) is established with K_1 being the maximum of the three sums on the right hand side of the above inequalities. To prove (2.22b) note that $\bar{\gamma}_k$ will have smaller value when $k\varepsilon\mathcal{S}_2,\mathcal{S}_3$ compared to its value in $\mathcal{S}_1$ for large value

of k and thus

$$\liminf_{k\to\infty} \gamma_k \geq \bar{r}_k^{-(1/2+\varepsilon)} [\min C,\ K^{-1/2}]$$

resulting in

$$\liminf_{k} \gamma_k \bar{r}_k^{(1/2+\varepsilon)} \geq K_2\ ,\ K_2 = \min [C,\ K^{-1/2}] > 0$$

Now for $k \in \mathscr{S}_1, \mathscr{S}_2$

$$\gamma_k \hat{x}_k' \hat{B}_{k-1} \hat{x}_k \leq \max [\bar{K}k^{-1},\ k^{-1/2} K^{1/2}]\ \bar{\gamma}_k^{-\varepsilon}$$

$$\leq K_3$$

where K_3 can be chosen such that $K_3 \geq \|\hat{B}_0\|$, thus established (2.22c) and completing the proof of lemma 2.1.

∇∇∇

<u>Lemma A1</u> For any arbitrary $\bar{\varepsilon} > 0$, with r_k defined as in (2.21d) one obtains

$$\sum_0^\infty \hat{x}_k' \hat{x}_k \ r_k^{-(1+\bar{\varepsilon})} < \infty \tag{A4}$$

<u>*Proof:*</u> The proof of the lemma is given in [28] and is presented here for easy reference. Select an integer m such that $\bar{\varepsilon} \geq \frac{1}{2^m}$ and denote $N \triangleq 2^m$, then the following algebraic manipulations yield the desired result

$$\sum_{k=0}^{\infty} \frac{\hat{x}_k' \hat{x}_k}{r_k^{(1+\bar{\varepsilon})}} \leq \sum_{k=0}^{\infty} \frac{\hat{x}_k' \hat{x}_k}{r_k^{(1+1/N)}}$$

$$= \sum_{k=0}^{\infty} \frac{(r_k - r_{k-1})}{r_k^{(1+1/N)}}$$

$$\leq \sum_{k=0}^{\infty} \frac{2^m r_k^{(1-1/N)} (r_k^{1/N} - r_{k-1}^{1/N})}{r_k^{1+1/N}}$$

$$\leq 2^m \sum_{k=0}^{\infty} \left(\frac{1}{r_k^{1/N}} - \frac{1}{r_{k-1}^{1/N}} \right)$$

$$\leq 2^m \, r_0^{-1/N} < \infty \, .$$

∇∇∇

REFERENCES

[1] Ledwich, G. and Moore, J.B., "Multivariable self-tuning filters", in Lecture Notes in Pure and Applied Mathematics, Differential Games and Control Theory II, Vol. 30, *Proc. Second Kingston Conference*, Kingston, June 1976, pp. 345-374.

[2] Moore, J.B. and Ledwich, G., "Multivariable adaptive parameter and state estimators with convergence analysis", *Journal of the Australian Mathematical Society*, Vol. 21, Part 2, pp. 176-197.

[3] Solo, V., "Time series recursions and stochastic approximation", Ph.D. dissertation, The Australian National University, September, 1978.

[4] Ljung, L., "On positive real transfer functions and the convergence of some recursive schemes", *IEEE Trans. Auto. Control*, Vol. AC-22, No. 4, August 1977, 539-551.

[5] Ljung, L., "Analysis of recursive stochastic algorithms", *IEEE Trans. Auto. Control*, Vol. AC-22, No. 4, August 1977, pp. 551-575.

[6] Goodwin, G.C., Ramadge, P.J. and Caines, P.E., "Discrete time stochastic adaptive control", *SIAM J. of Control and Optimization*, to appear, see also, *Proc. of 18th IEEE Conference on Decision and Control*, Vol. 2, December 1979, pp. 736-739.

[7] Feuer, A. and Morse, A.S., "Adaptive control of single-input single-output linear systems", *IEEE Trans. on Auto. Control*, Vol. AC-23, No. 1, pp. 532-538, August 1978.

[8] Egardt, B., "Stability of model reference adaptive and self tuning regulators", Department of Automatic Control, Lund Institute of Technology, December 1978.

[9] Goodwin, G.C. Ramadge, P.J. and Caines, P.E., "Discrete time multivariable adaptive control", *IEEE Trans. Auto. Control*, Vol. AC-25, pp. 449-456.

[10] Narendra, K.S. and Lin, Y.H., "Stable discrete adaptive control", S. & I.S. Report 7901, Yale University, March 1979.

[11] Astrom, K.J. and Wittermark, B., "On self-tuning regulators", *Automatica,* Vol. 9, 1973, pp. 195-199.

[12] Ljung, L. and Wittenmark, B., "Analysis of a class of adaptive regulators", *Proc. IFAC Symposium on Stochastic Control,* Budapest, Hungary, September 1974.

[13] Ljung, L. and Wittenmark, B., "On a stabilizing property of adaptive regulators", Reprint, *IFAC Symposium on Identification,* Tbilisi, U.S.S.R., 1976.

[14] Astrom, K.J., Borisson, U., Ljung, L. and Wittenmark, B., "Theory and application of self tuning regulators", *Automatica,* Vol. 10, 1977, pp. 457-476.

[15] Goodwin, G.C. and Sin, K.S., "Stochastic adaptive control using a modified least square algorithm", Technical Report EE7907, June 1979, University of Newcastle, N.S.W., Australia.

[16] Kumar, R. and Moore, J.B., "Convergence of adaptive minimum variance algorithms via weighting coefficient selection", Electrical Engineering Department, University of Newcastle, Tech. Report EE7917, 1979.

[17] Sin, K.S., Goodwin, G.C. and Bitmead, R.R., "An adaptive d-step ahead predictor based on least squares", *Proceedings of 19th IEEE Conference on Decision and Control,* Albuquerque, December 1980, pp. 962-967.

[18] Goodwin, G.C. and Long, R.S., "Generalizations of results on multivariable adaptive control", *Proceedings of 19th IEEE Conference on Decision and Control,* Albuquerque, December 1980, pp. 599-604.

[19] Moore, J.B. and Kumar, R., "Convergence of weighted minimum variance N-step ahead prediction/control schemes", *Proceedings of 19th IEEE Conference on Decision and Control,* Albuquerque, Dec. 1980, pp. 968-973.

[20] Desoer, C.A. and Vidyasagar, M. Feedback Systems: Input-Output Properties, Academic Press, New York, 1975.

[21] Hitz, K.L. and Anderson, B.D.O., "Discrete positive real functions and their applications in system stability", *Proc. IEE*, Vol. 116, 1969, pp. 153-155.

[22] Anderson, B.D.O. and Moore, J.B., "New results in linear system stability", *SIAM J. of Control*, Vol. 7, No. 3, August 1969.

[23] Neveu, J., Discrete Parameter Martingales, North-Holland Publishing Company, Amsterdam, 1975, p. 33.

[24] Sternby, J., "On consistency for the method of least squares using martingale theory", *IEEE Trans. on Auto. Control*, Vol. AC-22, No. 3, June, 1977, pp. 346-352.

[25] Kumar, R. and Moore, J.B., "Minimum variance control harnessed for non-minimum-phase plants", Electrical Engineering Department, University of Newcastle, Tech. Report EE80 , July 1980.

[26] Moore, J.B., "Persistence of excitation in extended least squares", Submitted for publication.

[27] Moore, J.B., "Convergence of output error recursions in colored noise", Submitted for publication.

[28] Kumar, R. and Kushner, H. J., "Almost sure convergence of adaptive prediction and control algorithms", submitted for publication.

[29] Kushner, H.J., "A projected stochastic approximation method for adaptive filters and indentifiers", *IEEE Trans. on Auto. Control*, August 1980, pp. 836-838.

[30] Kushner, H.J. and Kumar, R., "Almost sure convergence of adaptive prediction based on stochastic approximation algorithm with projections", submitted for publication.